Relativistic Effects and Terrestrial Clocks

Peter Lynch, UCD

November 2012

Abstract

Both special relativity (SR) and general relativity (GR) have consequences for the rate at which time passes. We consider the implications for clocks located on Earth at the north pole and equator, and find that, since the Earth's surface is an isopotential surface for apparent gravity, the effects of SR and GR cancel exactly.

Introduction

Consider two physically identical clocks, one located at the north pole and one at the equator. Since the clock at the equator is rotating relative to that at the pole, special relativity would indicate that it should tick at a slower rate. But, as the Earth is flattened, the clock at the equator is farther from the centre of the Earth and therefore in a weaker gravitational field. Thus, according to general relativity, it should tick at a faster rate. Remarkably, on the Earth's surface, these two effects cancel exactly, so the two clocks tick at the same rate.

The relativistic effects have practical consequences for the Global Positioning System. Indeed, GPS would be useless for navigation if the appropriate relativistic adjustments were not made.

The Spheroidal Earth

We will assume that the Earth is an oblate spheroid with equatorial radius a and polar radius b. The eccentricity e is given by $e^2 = (a^2 - b^2)/a^2$. Assuming small eccentricity, $a \approx b$, we have

$$e^2 = \left(1 - \frac{b}{a}\right)\left(1 + \frac{b}{a}\right) \approx 2a\left(\frac{1}{b} - \frac{1}{a}\right) . \tag{1}$$

The angular velocity of rotation is $\Omega = 2\pi$ radians per sidereal day. Let the acceleration due to (true) gravity be g^*_{EQ} at the equator and g^*_{NP} at the north pole. By Newton's Law of Gravity,

$$g^*_{\mathrm{EQ}} = \frac{GM}{a^2} \qquad \text{and} \qquad g^*_{\mathrm{NP}} = \frac{GM}{b^2} .$$

The centrifugal acceleration at the equator is $\Omega^2 a$. At the pole, it vanishes. Thus, the *apparent gravity* at the equator is $g_{\mathrm{EQ}} = g^*_{\mathrm{EQ}} - \Omega^2 a$ and at the pole it is $g_{\mathrm{NP}} = g^*_{\mathrm{NP}}$. The potential due to apparent gravity is

$$\Phi = -\frac{GM}{r} - \tfrac{1}{2}\Omega^2 R^2$$

where R is the distance from the polar axis. Thus, the values of potential at the pole and equator are

$$\Phi_{\mathrm{NP}} = -\frac{GM}{b} \qquad \text{and} \qquad \Phi_{\mathrm{EQ}} = -\frac{GM}{a} - \tfrac{1}{2}\Omega^2 a^2 .$$

Now let us assume that the Earth's surface is isopotential for apparent gravity, i.e., $\Phi_{\mathrm{NP}} = \Phi_{\mathrm{EQ}}$. Then

$$GM\left(\frac{1}{b} - \frac{1}{a}\right) = \tfrac{1}{2}\Omega^2 a^2 \tag{2}$$

which relates the rate of rotation to the Earth's eccentricity.

Special Relativity

Let $\mathrm{d}t_{\mathrm{EQ}}$ be a clock-tick at the equator and $\mathrm{d}t_{\mathrm{NP}}$ a clock-tick at the pole. In the "stationary frame" of the pole, we have

$$\mathrm{d}t_{\mathrm{EQ}}^2 = \left(1 - \frac{v^2}{c^2}\right) \mathrm{d}t_{\mathrm{NP}}^2 \,,$$

where v is the speed at the equator due to rotation and c is the speed of light. Since $v = \Omega a$, we have

$$\mathrm{d}t_{\mathrm{EQ}}^2 = \left(1 - \frac{\Omega^2 a^2}{c^2}\right) \mathrm{d}t_{\mathrm{NP}}^2 \,.$$

Let us define the time change due to special relativistic effects as

$$\Delta_{\mathrm{SR}} \equiv \left[\mathrm{d}t_{\mathrm{EQ}}^2 - \mathrm{d}t_{\mathrm{NP}}^2\right] = -\left(\frac{\Omega^2 a^2}{c^2}\right) \mathrm{d}t_{\mathrm{NP}}^2 \,. \tag{3}$$

General Relativity

According to general relativity, time is slowed down by a gravitational field. The clock-ticks at the equator and pole are given by

$$\mathrm{d}t_{\mathrm{EQ}}^2 = \left(1 - \frac{2GM}{c^2 a}\right) \mathrm{d}t_\infty^2 \,, \qquad \mathrm{d}t_{\mathrm{NP}}^2 = \left(1 - \frac{2GM}{c^2 b}\right) \mathrm{d}t_\infty^2 \,,$$

where $\mathrm{d}t_\infty$ is the clock-tick remote from mass. Let us define the time change between the equator and pole due to the effects of general relativity as

$$\Delta_{\mathrm{GR}} \equiv \left[\mathrm{d}t_{\mathrm{EQ}}^2 - \mathrm{d}t_{\mathrm{NP}}^2\right] = \left(\frac{2GM}{c^2 b} - \frac{2GM}{c^2 a}\right) \mathrm{d}t_\infty^2 \,.$$

Assuming that the corrections are small, we can replace $\mathrm{d}t_\infty$ by $\mathrm{d}t_{\mathrm{NP}}$ and write

$$\Delta_{\mathrm{GR}} = \frac{2GM}{c^2}\left(\frac{1}{b} - \frac{1}{a}\right) \mathrm{d}t_{\mathrm{NP}}^2 \,.$$

But, using (2), this can be written

$$\Delta_{\mathrm{GR}} = \left(\frac{\Omega^2 a^2}{c^2}\right) \mathrm{d}t_{\mathrm{NP}}^2 \,. \tag{4}$$

This is equal in magnitude but opposite in sign to the effect due to special relativity, so we have

$$\Delta_{\mathrm{SR}} + \Delta_{\mathrm{GR}} = 0 \,.$$

Conclusion

We have shown that the effects of SR and GR cancel in such a way that clocks at the equator and pole measure time at the same rate. We have not considered latitudes other than 0° and 90° or elevations other than sea-level.

The relativistic effects have practical consequences for the Global Positioning System. A GPS satellite is moving fast relative to the Earth's surface, but is also in a significantly weaker gravitational field. Theese effects partially cancel, but the latter is dominant, so the satellite clock tends to run fast. This is compensated by appropriate relativistic adjustments to the GPS clocks.