Joyce's Number

What is the largest number that can be written using only three decimal digits? An initial guess might be 999. But soon we realize that factorials permit much greater nummmbers such as 999! and exponents make even greater numbers possible.

The question is not really well-posed. To remove any ambiguity, we disallow the use of any symbols other than the three digits, and formulate the problem more precisely.

The Largest Three-digit Number

Problem: *What is the largest number that can be written in standard mathematical notation, using only three decimal digits and no other symbols?*

Since powers are written as superscripts, they are allowed. Clearly, we should use only the digit 9, and we can consider numbers like

$$99^9 \qquad (9^9)^9 = 9^{81} \qquad 9^{99} \qquad 9^{9^9}\,.$$

These are in increasing order, and the last one seems to be the greatest possible. In recognition of the role of James Joyce (see below), let us denote it $J = 9^{9^9}$.

But we can do much better. Donald Knuth introduced his up-arrow notation in 1976. Here $3 \uparrow 4$ is 3 to the 4th power, while $3 \uparrow\uparrow 4$ is $3 \uparrow (3 \uparrow (3 \uparrow 3))$. We can iterate this idea using any number of arrows. All but the first two are disallowed as solutions to our problem, since they involve additional symbols. However,

$$3 \uparrow 4 \ \text{ is written } \ 3^4 \quad \text{while} \quad 3 \uparrow\uparrow 4 \ \text{ is written } \ {}^4 3$$

So both qualify as potential solutions to the problem posed above. Of course, $3 \uparrow\uparrow 4$ is vastly greater than $3 \uparrow 4$.

If we accept Knuth's definition as "standard mathematical notation", the answer to the problem posed above seems to be the reverse triple power-tower $K = {}^{{}^9 9}9$, denoted K in recognition of Knuth.

Knuth's triple power-tower is breathtakingly large: Joyce's number J is dwarfed into insignifcance by K.

Joyce's Sequence

In the Ithaca episode of *Ulysses* (Chapter 17), Joyce wrote about Leopold Bloom contemplating a large number: "the 9th power of the 9th power of 9", and he indicated that to print the result would require "33 closely printed volumes of 1000 pages each".

The number Joyce specified (reading his text literally) was $G = (9^9)^9$. This number can easily be bounded above:

$$(9^9)^9 < (10^{10})^{10} = 10^{100} \equiv 1 \text{ googol}.$$

So, it is quite unremarkable (is it the *smallest* unremarkable number?!).

However, we must make allowances: Joyce's performance in his school and university mathematics ranged from indifferent to abysmal. It is clear that he really meant "9 to the 9th power of 9", or 9^{9^9}, which is much larger than G. We use the symbol J to denote the number 9^{9^9}.

The number $J = 9^{9^9}$ can be bounded below. Without resorting to a calculator, and using only the result $2^{10} > 10^3$, we have

$$\begin{aligned} 9^9 &= 9 \times 9^8 = 9 \times 81^4 > 9 \times 80^4 > 9 \times 8^4 \times 10^4 = 9 \times 2^{12} \times 10^4 \\ &= 9 \times 4 \times 2^{10} \times 10^4 > 9 \times 4 \times 10^3 \times 10^4 = 36 \times 10^3 \times 10^4 \\ &= 3.6 \times 10^8 \end{aligned}$$

So, the power to which 9 is raised in the triple power-tower J is about 360 million. Using this result in J, we have

$$\begin{aligned} 9^{9^9} &> 9^{3.6\times10^8} = 81^{1.8\times10^8} = 8^{1.8\times10^8} \times 10^{1.8\times10^8} = 2^{5.4\times10^8} \times 10^{1.8\times10^8} \\ &= (2^{10})^{5.4\times10^7} \times 10^{1.8\times10^8} > (10^3)^{5.4\times10^7} \times 10^{1.8\times10^8} = 10^{1.62\times10^8} \times 10^{1.8\times10^8} \\ &= 10^{3.42\times10^8} \end{aligned}$$

This confirms that the triple power-tower J has more than 340 million digits.

A weighty tome

How many volumes would be required to print Joyce's number?

On the MathWorld site, the sequence $\{J_n\}$ giving the number of digits in each term of the threefold power-tower sequence $\{n^{n^n}\}$ is called the Joyce sequence. The values of the first few entries in the power-tower sequence are given in Slone's Online Encyclopedia of Integer Sequences (OEIS) A002488. They grow very fast:

$$0, 1, 16, 7625597484987, \ldots$$

Joyce's sequence is the number of digits in each term of this sequence, given in A054382. The first eleven entries are:

$$1, 1, 2, 13, 155, 2185, 36306, 695975, 15151336, 369693100, 10000000001.$$

Figure 1: Joyce's Tower in Sandycove, Co. Dublin [Photo P Lynch]

So, the number of digits in $J = 9^{9^9}$ is 369,693,100. Our estimate above, based on a lower bound of J, was 342,000,000 which is close enough.

To see Joyce's number in all its glory, we need to print about 370 million digits. Assuming 100 digits per line and 100 lines per page, this implies that something like 37 volumes, each of 1000 pages, would be required to write down J explicitly. Joyce's estimate in *Ulysses* was 33 volumes; not bad, considering his dismal performance in mathematics.

Sources

Bloomsday page on UCD Library site: http://www.ucd.ie/library/news_publicity/bloomsday2013/

Knuth's up-arrow notation is here: http://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation.

The Joyce sequence is defined on MathWorld: http://mathworld.wolfram.com/JoyceSequence.html

Online Encyclopedia of Integer Sequences: A002488 http://oeis.org/A002488

Online Encyclopedia of Integer Sequences: A054382 http://oeis.org/A054382