Curvature of Poincaré's Half-plane Model

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We will consider Poincaré's half-plane model for hyperbolic geometry in two dimensions. It is named after Henri Poincaré who studied it intensively, although it was originally formulated by Eugenio Beltrami as a model for non-Euclidean geometry.

The half-plane model comprises the upper half plane $H = \{(x, y) : y > 0\}$ together with a metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}\,.$$

It is remarkable that the entire structure of the space follows from the metric, although not without some effort.

Metric and Geodesics

What are the "straight lines" in this model? For two points P and Q in H, the distance between them is

$$s = \int_P^Q \mathrm{d}s = \int_P^Q \sqrt{\frac{1 + y'^2}{y^2}}\,\mathrm{d}x = \int_P^Q L(y, y')\,\mathrm{d}x$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $L(y, y')$ may be called the Lagrangian. We can write down the Euler-Lagrange equations for the solution that minimises this distance:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0 \qquad \text{becomes} \qquad yy'' + (1 + {y'}^2) = 0\,.$$

This is the equation for the geodesics $y(x)$.

How to Solve the Geodesic Equation

One way to find the solution is to look it up in a dusty old book like Kamke [2], where the equation is found on page 573 (§6.126). The solutions are described as *Halbkreise*, or semicircles. So, let's try the general equation for a circle

$$(x - x_0)^2 + (y - y_0)^2 = a^2\,.$$

Plugging this into the equation, we find that it is a solution if $y_0 = 0$. Thank you very much, Herr Dr. Kamke [2].

Thus, the geodesics in H are the semicircles

$$y = \sqrt{a^2 - (x - x_0)^2}$$

with centre at $(x_0, 0)$ and radius a. Some of these are shown in the figure below.

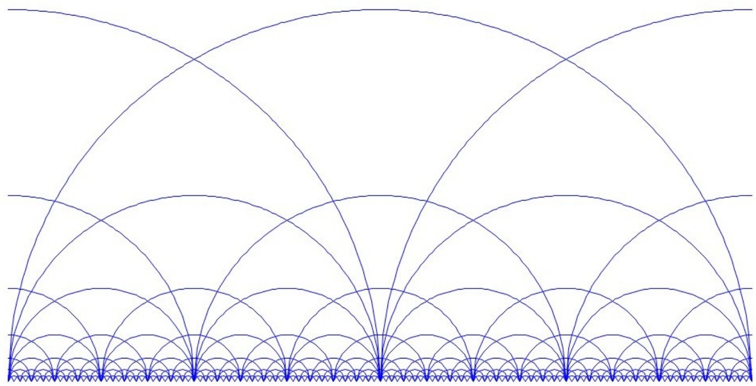

Figure 1: Geodesics in the Poincaré Half-plane

Why Hyperbolic Geometry?

We can compute the curvature of the half-plane $(H, \mathrm{d}s)$. The details are many, and we shall show only the main steps. We write the metric, with the usual summation convention, as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu x^\nu$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix} \qquad \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

The Christoffel symbols of the first and second kind are

$$[\alpha\,\beta, \gamma] = \tfrac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) \qquad \Gamma^\gamma_{\alpha\beta} = g^{\gamma\delta}[\alpha\,\beta, \delta]$$

There are eight of each kind but, for the half-plane, they are zero with the following exceptions:

$$[1\,2, 1] = [2\,1, 1] = -[1\,1, 2] = [2\,2, 2] = -1/y^3$$

$$\Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} = -1/y$$

Now we substitute these into the Riemann curvature tensor:

$$R^\sigma{}_{\alpha\beta\gamma} = \frac{\partial\Gamma^\sigma_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial\Gamma^\sigma_{\beta\gamma}}{\partial x^\alpha} + \Gamma^\tau_{\alpha\gamma}\Gamma^\sigma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\gamma}$$

See Fleisch [1] for more details. There are sixteen components in this tensor and, once again, the algebra is intricate. But most of the components vanish, and we are left with just

$$R^1{}_{212} = -R^1{}_{221} = R^2{}_{121} = -R^2{}_{112} = -1/y^2$$

In fully covariant form, the non-zero components are

$$R_{1212} = R_{2121} = -1/y^4 \qquad R_{1221} = R_{2112} = +1/y^4$$

The Ricci tensor is the contraction

$$R_{\alpha\gamma} = R^{\sigma}{}_{\alpha\sigma\gamma}$$

All that survives are two components

$$R_{11} = R_{22} = -1/y^2$$

Finally, the Ricci scalar is the contraction

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu} = y^2(-1/y^2) + y^2(-1/y^2) = -2$$

In the two-dimensional case that we consider, this is related directly to the Gaussian curvature: $\kappa = \frac{1}{2}R$, so at last we have

$$\kappa = -1\,.$$

The Gaussian curvature of our half-plane model has a constant value -1. The space has uniform negative curvature and is a hyperbolic space.

A slight short-circuit is possible [3]. Defining $g = \det(g_{\mu\nu})$, we can write the Gaussian curvature in terms of an element of the Riemann tensor:

$$\kappa = R_{1212}/g = (-1/y^4)/(1/y^4) = -1\,.$$

An Easier Way to Evaluate Curvature

We define the principal curvatures k_1 and k_2 of a surface S as the maximum and minimum values of the curvature of a curve formed by the intersection of S with planes containing the normal to the surface. Clearly, k_1 and k_2 require measurements external to the surface itself; they are *extrinsic* quantities. In a remarkable theorem, the *Theorema Egregium*, Gauss showed that the total curvature $\kappa = k_1 k_2$ can be measured *intrinsically*, using only quantities that can be measured within the surface itself.

We assume that the surface is specified in terms of two parameters u and v as

$$x = x(u,v)\,, \qquad y = y(u,v)\,, \qquad z = z(u,v)\,.$$

The First Fundamental Form is

$$ds^2 = E(u,v)du^2 + 2F(u,v)du\,dv + G(u,v)dv^2.$$

where the functions E, F and G can be evaluated within the surface. Then the total curvature may be expressed in terms of these three functions and their first and second derivatives [3, pg. 183]. In the case of orthogonal coordinates, $F = 0$, we get:

$$\kappa = -\frac{1}{EG}\left[\frac{1}{2}(E_{vv} + G_{uu}) - \frac{1}{4}\left(\frac{E_u G_u + E_v^2}{E} + \frac{E_v G_v + G_u^2}{G}\right)\right]$$

For the present case the First Fundamental Form is

$$ds^2 = \frac{du^2 + dv^2}{v^2}$$

so we have $E = G = 1/v^2$ and $F = 0$. Then the expression for κ can be evaluated and yields the result:

$$\kappa = -1\,.$$

This is certainly simpler and more direct than evaluating the entire fourth-order Riemann tensor.

Many parallels through a Given Point

We can see from the figure of the half-plane, and the knowledge that the geodesics are semicircles with centres on the x-axis, that for a given "straight line" and a point not on it, there is more than one line that does not intersect the given line. That is, many lines can be drawn through the point that are parallel to the given line. An example is given in the figure below: both thin lines through P are parallel to the thick line. In fact, there are an infinite number of such lines.

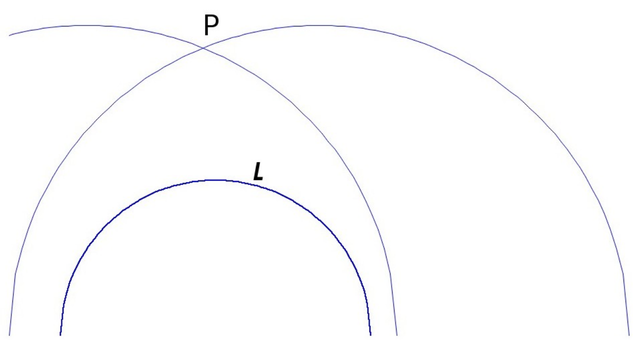

Figure 2: Both lines through P are parallel to the heavy line. There are an infinite number of such lines through P parallel to the heavy line.

Sources

[1] Fleisch, Daniel, 2012: *A Student's Guide to Vectors and Tensors.* Cambridge Univ. Press, 297pp.

[2] Kamke, E., 1948: *Differentialgleichungen: Lösungsmethoden und Lösungen. Band 1. Gewönliche Differentialgleichungen.* 3rd Edn., Chelsea Publ., New York. 666pp.

[3] Lanczos, Cornelius, 1979: *Space through the ages: The evolution of geometric ideas from Pythagoras to Hilbert and Einstein.* Academic Press. ISBN 0124358500.